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The mild-slope equations

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In its original form the mild-slope equation, which approximates the motion of linear water waves over undulating topography, is a simplified version of the more recently derived modified mild-slope equation. However, the reduced equation does not deal adequately with rapidly varying small-amplitude perturbations about an otherwise slowly varying bedform and it does not produce free-surface profiles that inherit slope discontinuities from the topography, an intrinsic feature of the approximation on which both equations are based. The inconsistency between the two equations is rectified by the derivation of an alternative form of the mild-slope equation, having the simplicity of the standard form and yet containing all of the essential features of the full equation. In the process, a more transparent version of the modified mild-slope equation is identified. The standard and revised mild-slope equations are compared analytically in the context of two-dimensional plane wave scattering and it is found that they lead to values of the reflected wave amplitude that differ at lowest order in the mild-slope parameter, for a general topography. It is also confirmed that the revised mild-slope equation gives the dominant contribution in the solution of the new form of the modified mild-slope equation. Indeed, the two equations differ only by a term that is virtually negligible.

1. Introduction

The mild-slope equations simplify the linearized scattering of surface waves on water of variable depth by approximating the vertical structure of the motion and averaging over the depth. In its original form, the mild-slope equation was derived independently by Berkhoff (1972, 1976) and Smith & Sprinks (1975), although it may also be found in Jonsson & Brink-Kjaer (1973) and Lozano & Meyer (1976). Among other derivations that have been given since are those by Kirby (1986), Miles (1991) and Chamberlain & Porter (1995). Miles & Chamberlain (1998) also showed that the mild-slope equation and its modified counterpart, given by Chamberlain & Porter (1995), arise in a systematically derived sequence of approximations to scattering over an uneven bed. Athanassoulis & Belibassakis (1999) have recently given a comprehensive survey of surface wave scattering by topography in two dimensions that includes multi-mode extensions of the mild-slope equation. Such extensions are not the immediate concern of the present investigation but they may benefit from a similar analysis to that carried out here.

To set the mild-slope equations in context, let x and y denote Cartesian coordinates lying in the undisturbed free surface $z=0$ and let the bed be situated at $z=-h(x,y)$. The continuous function $h(x,y)$ is assumed to satisfy the mild-slope approximation

$$|\nabla h/kh| = O(\epsilon) \quad (\epsilon \ll 1), \tag{1.1}$$

for all x, y, the local wavenumber $k = k(h(x, y))$ being the positive root of the dispersion relation

$$\nu \equiv \sigma^2/g = k \tanh(kh) \tag{1.2}$$

corresponding to the depth $h(x, y)$. The angular frequency σ is assumed to be given.

The condition (1.1) is satisfied by the bedform

$$h(x, y) = h_1(x, y) + h_2(x, y), \quad \text{where} \quad h_1(x, y) = H(\epsilon x, \epsilon y), \quad h_2(x, y) = \epsilon\, d(x, y). \tag{1.3}$$

Supposing that H and its derivatives are $O(1)$, the slowly varying profile $h_1(x, y)$ induces depth variations $O(1)$ over distances $O(1/\epsilon)$. With $d = O(1)$, the term $h_2(x, y)$ superposes rapid, small-amplitude variations on the slowly varying component.

The mild-slope equation results from approximating the velocity potential for the motion by

$$\Phi(x, y, z, t) \approx \mathrm{Re}\left\{ \frac{g}{\mathrm{i}\sigma} w_0(z, h)\, \eta(x, y) \mathrm{e}^{-\mathrm{i}\sigma t} \right\}, \quad w_0(z, h) = \mathrm{sech}(kh) \cosh k(z + h), \tag{1.4}$$

with $k(h)$ determined by (1.2), the scaling having been arranged so that the free-surface elevation is approximated by $\mathrm{Re}\left\{\eta(x, y)\mathrm{e}^{-\mathrm{i}\sigma t}\right\}$, where η is required to be continuous. The essence of the approximation is therefore that the vertical motion at x, y is taken to be that of a plane wave propagating on water having the local depth $h(x, y)$.

By implementing vertical averaging and in the process discarding small terms, the mild-slope equation was presented by Berkhoff (1972, 1976) and others in the form

$$\nabla \cdot u_0 \nabla \eta + k^2 u_0 \eta = 0, \tag{1.5}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$ and

$$u_0 = u_0(h) = \|w_0\|^2 = \{2kh + \sinh(2kh)\}/4k \cosh^2(kh). \tag{1.6}$$

The notation

$$(u, v) = \int_{-h}^{0} u(z, h) v(z, h)\, \mathrm{d}z, \quad \|u\|^2 = (u, u),$$

is a convenient abbreviation in the following account.

Chamberlain & Porter (1995) formalized the vertical averaging procedure by invoking a variational principle and, using the same approximation (1.4) for the velocity potential, derived the modified mild-slope equation

$$\nabla \cdot u_0 \nabla \eta + \{k^2 u_0 + u_1 \nabla^2 h + u_2 (\nabla h)^2\} \eta = 0, \tag{1.7}$$

in which

$$u_1(h) = (w_0, \dot{w}_0), \quad u_2(h) = \dot{u}_1(h) - \|\dot{w}_0\|^2, \tag{1.8}$$

the dot denoting differentiation with respect to h.

Equation (1.7) may be regarded as the 'complete' mild-slope equation, as it is derived solely on the basis of (1.4) without further approximation, allowing the effects of the terms with coefficients u_1 and u_2, implicitly discarded in earlier derivations, to be assessed. These terms are $O(\epsilon^2)$ for the slowly varying bed profile $h = h_1$, but $\nabla^2 h_2$ is $O(\epsilon)$, which is comparable in magnitude with that of the term $\nabla u_0 = \dot{u}_0 \nabla h$ in (1.5). This fact explains the failure of the mild-slope equation to give an accurate value of Bragg resonance in ripple bed scattering, as observed by Kirby (1986) who amended the equation by including a rapidly varying small-amplitude term.

However, numerical results have shown that discrepancies between the solutions of (1.5) and (1.7) which are disproportionate to the magnitude of the additional terms in the latter are not confined to the special case of ripple beds. There is a further source of error that is attributable to the $\nabla^2 h$ term in (1.7), which implies discontinuities in $\nabla\eta$. This follows by direct integration of the equation to give

$$u_0[\boldsymbol{n}\cdot\nabla\eta] + u_1\eta\,[\boldsymbol{n}\cdot\nabla h] = 0, \tag{1.9}$$

where [] denotes the jump in the included quantity across a line of discontinuity of ∇h and $\boldsymbol{n}$ is a unit normal to that line. A slope discontinuity in the free-surface profile induced by a slope discontinuity in the topography is therefore inherent in the approximation, but it does not arise from (1.5) which implies that $\nabla\eta$ is continuous everywhere. Numerical experiments (see, for example, Porter & Staziker 1995) suggest that the failure of (1.5) to model such discontinuities may be significant.

The purpose of this paper is to re-examine equations (1.5) and (1.7) and resolve the discrepancies between their solutions. This is achieved in the next section by transforming (1.7) in such a way that a reduced form of it, a new version of the mild-slope equation, contains all of the essential characteristics of the full modified mild-slope equation. A comparison of the existing and alternative forms of the mild-slope equation is carried out in § 3 for the most straightforward application, to two-dimensional scattering, using an analytic solution method which may be applied for any depth function.

2. Alternative forms of the mild-slope equations

We transform (1.7) by setting

$$\eta(x, y) = s(h)\zeta(x, y), \tag{2.1}$$

where the scaling $s(h(x, y))$ is to be determined. Since

$$\nabla\cdot u_0\nabla(s\zeta) = u_0 s\nabla^2\zeta + \{(u_0 s)^{\cdot} + u_0\dot{s}\}\nabla h\cdot\nabla\zeta + \{u_0\dot{s}\nabla^2 h + (u_0\dot{s})^{\cdot}(\nabla h)^2\}\eta,$$

(1.7) implies that

$$u_0 s\nabla^2\zeta + \{(u_0 s)^{\cdot} + u_0\dot{s}\}\nabla h\cdot\nabla\zeta + \{k^2 u_0 s + \widetilde{u}_1\nabla^2 h + \widetilde{u}_2(\nabla h)^2\}\zeta = 0, \tag{2.2}$$

where

$$\widetilde{u}_1 = u_1 s + u_0\dot{s}, \quad \widetilde{u}_2 = u_2 s + (u_0\dot{s})^{\cdot}.$$

Selecting s so that $(u_0 s)^{\cdot} + u_0\dot{s} = 0$ gives the standard canonical form of the modified mild-slope equation. Here we make the different choice $\widetilde{u}_1 = 0$ to remove the term $\nabla^2 h$, which we have noted to be a source of discrepancies between the solutions of (1.5) and (1.7).

The dispersion relation implies that $\dot{k} = -2k^2\{2kh + \sinh(2kh)\}^{-1}$ from which the identity

$$2u_1 = \dot{u}_0 + 2u_0\dot{k}/k \tag{2.3}$$

may readily be verified. This relationship allows $\widetilde{u}_1 = 0$ to be integrated at once to give $u_0^{1/2}ks = \text{constant}$ and we take

$$s(h) = 1/k(h)\{u_0(h)\}^{1/2}.$$

Moreover, $\widetilde{u}_1 = 0$ used with (1.6) and (1.8) implies that

$$\widetilde{u}_2 = \{\dot{u}_1 - \|\dot{w}_0\|^2\}s - (u_1 s)^{\cdot} = -s\{\|\dot{w}_0\|^2 - (w_0, \dot{w}_0)^2/\|w_0\|^2\}$$

and with (2.3) it shows that

$$(u_0 s)^{\cdot} + u_0 \dot{s} = -2u_0 s\dot{k}/k.$$

The result of using these various expressions in (2.2) is the self-adjoint equation

$$\nabla \cdot k^{-2}\nabla\zeta + \{1 - v\,(\nabla h)^2\}\zeta = 0, \tag{2.4}$$

where the dimensionless coefficient v is given by

$$v(h) = \{\|w_0\|^2\|\dot{w}_0\|^2 - (w_0, \dot{w}_0)^2\}/k^2\|w_0\|^4. \tag{2.5}$$

The spatial component of the free-surface elevation is determined from the solution ζ of (2.4) through the transformation (2.1), which may be expressed explicitly as

$$\eta(x, y) = \frac{2\cosh(kh)\,\zeta(x, y)}{(k(2kh + \sinh(2kh))^{1/2}}. \tag{2.6}$$

The function $v(h)$ has to be evaluated for practical purposes, of course. We note that it is defined by an expression that is invariant under a scaling of w_0 by an arbitrary function of h; that is, $v(h)$ is unchanged if w_0 is replaced by $c(h)w_0$ with $\dot{c} \neq 0$. Therefore the evaluation can be made by using just $w_0(z, h) = \cosh k(z + h)$, leading to

$$v(h) = \{3(2K + \sinh K)(\sinh(2K) - \sinh K) - 3K^2(\cosh(2K) + 2) \\ - 4K^3\sinh K - K^4\}/3(K + \sinh K)^4,$$

where $K = 2kh$, an expression which belies the simple form of (2.5).

The transformed version (2.4) of the modified mild-slope equation has a number of advantages over the original form (1.7). Its relatively simple structure allows properties of the solution to be deduced more easily; the conclusions drawn by Chamberlain & Porter (1996), for example, can be simplified and extended by using (2.4). The removal of the term involving $\nabla^2 h$ both eliminates an $O(\epsilon)$ term and means that $\nabla\zeta$ is continuous everywhere, a practical asset when analysing the equation and computing its solutions; the discontinuity in $\nabla\eta$ remains, of course, and arises through (2.6). Further, the $O(\epsilon^2)$ term in (2.4) is sign definite since the Cauchy–Schwarz inequality applied to (2.5) shows that $v(h) \geqslant 0$. This is significant when deducing qualitative differences produced by the higher-order term.

We also note that (2.4) is a direct extension to general kh of the shallow-water equation $\nabla \cdot h\nabla\zeta + \nu\zeta = 0$, which may be written as

$$\nabla \cdot k^{-2}\nabla\zeta + \zeta = 0, \tag{2.7}$$

on using the shallow-water version $\nu = k^2 h$ of (1.2). Since $\eta \to \zeta/\nu^{1/2}$ and $v(h) = O((kh)^2) \to 0$ as $kh \to 0$, (2.7) is indeed the shallow-water limit of (2.4).

More significantly, we may regard the truncated form (2.7) of (2.4) as a 'consistent' version of the mild-slope equation for any value of kh in that it is correct to $O(\epsilon^2)$, as was envisaged but incorrectly implemented in the original derivations. Moreover, (2.7) is actually simpler than the standard mild-slope equation (1.5) and it leads to approximations of the free-surface profile that satisfy the jump condition (1.9) implicit in the mild-slope approximation, by virtue of (2.6).

The modified mild-slope equation will clearly produce the most accurate solutions and the transformed version (2.4) simplifies its use. The main interest therefore centres on the relative merits of the usual mild-slope equation (1.5) and its new alternative form (2.7).

The ratio u_0k^2 of the corresponding coefficients in (1.5) and (2.7) is a decreasing function of kh and satisfies

$$\nu/2 < u_0k^2 = s^{-2} \leqslant \nu,$$

the upper limit being attained in the shallow-water case. To determine how this variation affects the solutions of the respective equations, we examine a specific problem in which the differences can be analysed.

3. Two-dimensional scattering

We disregard the coordinate y and consider the equation

$$(p\chi')' + pk^2(1-q)\chi = 0, \tag{3.1}$$

holding for $-\infty < x < \infty$, which can be aligned with the one-dimensional forms of (1.5), (2.4) and (2.7) by appropriate choices of χ, p and q. As already noted, it is consistent with those equations to seek solutions for χ that are continuous and have continuous derivatives for all x.

The bed is assumed to be continuous and horizontal outside the interval $(0, \ell)$, with

$$h(x) = \begin{cases} h_0 & (x \leqslant 0) \\ h_1 & (x \geqslant \ell), \end{cases}$$

where h_0 and h_1 are constants. Taking into account the time dependence in (1.4) and supposing that a wave of unit amplitude is incident from the left, we set

$$\chi(x) = \begin{cases} \mathrm{e}^{\mathrm{i}k_0x} + R\mathrm{e}^{-\mathrm{i}k_0x} & (x \leqslant 0) \\ T\mathrm{e}^{\mathrm{i}k_1(x-\ell)} & (x \geqslant \ell), \end{cases} \tag{3.2}$$

where $k_0 = k(h_0)$ and $k_1 = k(h_1)$.

Because of the required continuity, boundary conditions for (3.1) on $(0, \ell)$ follow from (3.2) as

$$\chi'(0) + \mathrm{i}k_0\chi(0) = 2\mathrm{i}k_0, \quad \chi'(\ell) - \mathrm{i}k_1\chi(\ell) = 0, \tag{3.3}$$

and the equations

$$\chi'(0) - \mathrm{i}k_0\chi(0) = -2\mathrm{i}k_0R, \quad \chi'(\ell) + \mathrm{i}k_1\chi(\ell) = 2\mathrm{i}k_1T \tag{3.4}$$

recover the complex amplitudes R and T of the reflected and transmitted waves, which may be regarded as the principal unknowns of the problem.

Numerical solutions of (3.1) and (3.3) may readily be obtained for any given depth function $h(x)$ in $(0, \ell)$. However, an analytic solution is developed here as the objective is to identify the differences between the approximations resulting from the various forms of the mild-slope equations for a general bedform $h(x)$.

3.1. *An analytic solution*

An integral equation method is used to solve the boundary value problem consisting of (3.1) and (3.3). This approach avoids complications arising from the finite jumps that may occur in the term h'^2 when the application to (2.4) is considered.

To simplify the integration step, we first cast (3.1) in a different form by introducing the continuous functions $\phi_1(x)$ and $\phi_2(x)$ where

$$\chi' + \mathrm{i}k\chi = 2\mathrm{i}k\phi_1, \quad \chi' - \mathrm{i}k\chi = -2\mathrm{i}k\phi_2 \quad (0 \leqslant x \leqslant \ell),$$

in which $k = k(h(x))$. Since $\chi = \phi_1 + \phi_2$ and $\chi' = \mathrm{i}k(\phi_1 - \phi_2)$, we must have

$$\phi_1' + \phi_2' = \mathrm{i}k(\phi_1 - \phi_2)$$

and (3.1) implies that

$$pk(\phi_1' - \phi_2') = -(pk)'(\phi_1 - \phi_2) + \mathrm{i}k(1 - q)(\phi_1 + \phi_2).$$

Solving for ϕ_1' and ϕ_2' and using an integrating factor reduces these two equations to the forms

$$(\phi_1(x)/\alpha(x))' = -\overline{b(x)}\phi_2(x)/\alpha(x), \quad (\phi_2(x)/\overline{\alpha(x)})' = -b(x)\phi_1(x)/\overline{\alpha(x)} \quad (0 < x < \ell), \tag{3.5}$$

in which

$$\alpha(x) = \exp\left(\int_0^x a(s)\,\mathrm{d}s\right), \quad a = \mathrm{i}k - \frac{1}{2}\left(\frac{(pk)'}{pk} + \mathrm{i}kq\right), \quad b = a - \mathrm{i}k. \tag{3.6}$$

Further, (3.3) and (3.4) imply that

$$\phi_1(0) = 1, \quad \phi_2(\ell) = 0, \quad \phi_1(\ell) = T, \quad \phi_2(0) = R, \tag{3.7}$$

the first two elements serving as boundary conditions for (3.5).

Now if $(\psi_1, \psi_2)^T$ is a solution of the coupled system (3.5) so is $(\overline{\psi_2}, \overline{\psi_1})^T$ and this pair is linearly independent provided that $|\psi_1(0)| \neq |\psi_2(0)|$ since its Wronskian is equal to

$$|\psi_1(x)|^2 - |\psi_2(x)|^2 = (|\psi_1(0)|^2 - |\psi_2(0)|^2)|\alpha(x)|^2.$$

We choose

$$\psi_1(0) = 0, \quad \psi_2(0) = 1 \tag{3.8}$$

(so that $\psi_2(x)$ is non-vanishing) and the solution of (3.5) may therefore be written as $(\phi_1, \phi_2)^T = c_1(\psi_1, \psi_2)^T + c_2(\overline{\psi_2}, \overline{\psi_1})^T$ for some constants c_1 and c_2. Applying (3.7) we readily find that

$$R = -\overline{\psi_1(\ell)}/\psi_2(\ell), \quad T = |\alpha(\ell)|^2/\psi_2(\ell). \tag{3.9}$$

It remains to determine ψ_1 and ψ_2. Integration of (3.5) applied to these functions is immediate and using (3.8) it gives

$$\psi_1(x) = -\int_0^x \frac{\alpha(x)\overline{b(s)}}{\alpha(s)}\psi_2(s)\,\mathrm{d}s, \quad \psi_2(x) = \overline{\alpha(x)} - \int_0^x \frac{\overline{\alpha(x)}b(s)}{\overline{\alpha(s)}}\psi_1(s)\,\mathrm{d}s \quad (0 \leqslant x \leqslant \ell).$$

To solve this pair of coupled integral equations efficiently, it is convenient to define operators U and V on $L_2(0, \ell)$ by

$$(\mathrm{U}\psi)(x) = \overline{\psi(x)}, \quad (\mathrm{V}\psi)(x) = \int_0^x v(x,t)\psi(t)\,\mathrm{d}t,$$
$$v(x,t) = \overline{\alpha(x)}b(t)/\overline{\alpha(t)} \quad (0 \leqslant x, t \leqslant \ell),$$

in terms of which we have

$$\psi_1 = -\mathrm{UVU}\psi_2, \quad \psi_2 = \mathrm{U}\alpha - \mathrm{V}\psi_1. \tag{3.10}$$

Eliminating ψ_1,

$$\psi_2 = \mathrm{U}\alpha + (\mathrm{VU})^2\psi_2$$

and since $(\mathrm{VU})^2$ is a linear operator (whereas VU is not), the unique solution is given by the Neumann series

$$\psi_2 = \sum_{n=0}^{\infty} (\mathrm{VU})^{2n} \mathrm{U}\alpha,$$

from which

$$\psi_1 = -\mathrm{U} \sum_{n=0}^{\infty} (\mathrm{VU})^{2n+1} \mathrm{U}\alpha$$

follows at once by (3.10). Reference to Porter & Stirling (1990) shows that it is sufficient that $v(x,t)$ be a bounded function on $[0,\ell] \times [0,\ell]$ for the corresponding series of functions to converge uniformly for $x \in [0,\ell]$. This is certainly the case in the present application, in which h'^2 is bounded and all of the other functions involved are continuous.

To translate the solutions for ψ_1 and ψ_2 into practical forms we note that successive applications of the operator VU easily lead to the formula

$$(\mathrm{VU})^{2n} U\alpha = \overline{\alpha}\, \beta_n \quad (n = 0, 1, \ldots),$$

where the sequence $\{\beta_n\}$ is determined by the recurrence relation

$$\beta_0(x) = 1, \quad \beta_n(x) = \int_0^x g(t)\overline{\beta_{n-1}(t)}\, \mathrm{d}t \quad (n \in I\!N), \quad g(t) = \alpha(t)b(t)/\overline{\alpha(t)}. \tag{3.11}$$

Returning to (3.9), therefore, we find explicit expressions for the scattering coefficients R and T in the forms

$$T = \alpha(\ell) \Bigg/ \sum_{n=0}^{\infty} \beta_{2n}(\ell), \quad R = \sum_{n=0}^{\infty} \beta_{2n+1}(\ell) \Bigg/ \sum_{n=0}^{\infty} \beta_{2n}(\ell). \tag{3.12}$$

Since it can be shown by using (3.11) that

$$\left| \sum_{n=0}^{\infty} \beta_{2n}(x) \right|^2 = 1 + \left| \sum_{n=0}^{\infty} \beta_{2n+1}(x) \right|^2,$$

the real reflected wave amplitude can be written as

$$|R|^2 = \left| \sum_{n=0}^{\infty} \beta_{2n+1}(\ell) \right|^2 \Bigg/ \left(1 + \left| \sum_{n=0}^{\infty} \beta_{2n+1}(\ell) \right|^2 \right). \tag{3.13}$$

3.2. *The mild-slope equations*

We may now return to the main objective and use these results to compare the two versions (1.5) and (2.7) of the mild-slope equation in the context of two-dimensional scattering.

Comparing (1.5) with (3.1) we have $p = u_0$, $q = 0$ and $\chi \equiv \eta$. The alignment of (2.7) with (3.1) requires $p = k^{-2}$, $q = 0$ and $\chi \equiv \zeta = \eta/s(h)$ but to preserve the form (3.2) we introduce a constant multiplier and set $\eta = s(h)\zeta/s(h_0)$. It follows that the time-independent part of the free-surface profile is such that

$$\eta(x) = \begin{cases} \mathrm{e}^{\mathrm{i}k_0 x} + R\mathrm{e}^{-\mathrm{i}k_0 x} & (x \leqslant 0) \\ \widetilde{T}\mathrm{e}^{\mathrm{i}k_1(x-\ell)} & (x \geqslant \ell), \end{cases}$$

where $\widetilde{T} = T$ for (3.1) and $\widetilde{T} = s(h_1)T/s(h_0)$ for (2.7), R and T being determined by (3.12) applied to the relevant equation. In particular, the multiplier $\alpha(\ell)$ occurring in the expression for T reconciles the different scalings in $\widetilde{T}$.

We shall use the superscript 1 to refer to (1.5) and the superscript 2 to refer to (2.7). Thus, for example, on evaluating the generating functions (3.11) for the two equations by using the appropriate functions p and q and referring to (3.6) we find that

$$g^{(i)}(x) = -k(h(x))h'(x)f^{(i)}(K(x))\exp\left(2\mathrm{i}\int_0^x k(h(s))\,\mathrm{d}s\right) \quad (0 \leqslant x \leqslant \ell), \tag{3.14}$$

where $K = 2kh$, as before, and

$$f^{(1)}(K) = (2\sinh K - K\cosh K + K)/(K + \sinh K)^2, \quad f^{(2)}(K) = 1/(K + \sinh K).$$

Because of the multiplier h' in $g^{(i)}$, the sequences generated by using (3.14) in (3.11) are such that $\beta_n^{(i)} = O(\epsilon^n)$ for the general bedform (1.3). Few terms are therefore needed in (3.13) to obtain good estimates of the reflected wave amplitudes $|R^{(i)}|$ and for our present purpose we can restrict attention to the terms

$$\beta_1^{(i)}(\ell) = \int_0^\ell g^{(i)}(x)\,\mathrm{d}x = -\int_0^\ell k(h(x))h'(x)f^{(i)}(K(x))\exp\left(2\mathrm{i}\int_0^x k(h(s))\,\mathrm{d}s\right)\mathrm{d}x. \tag{3.15}$$

The modified mild-slope equation in the form (2.4), which corresponds to taking $p = k^{-2}$ and $q = vh'^2$ in (3.1), leads to an $O(\epsilon)$ correction in $f^{(2)}(K)$ which does not contribute to the dominant term in $\beta_1^{(2)}(\ell)$. This confirms that the amplitude $|R^{(2)}|$ given by (2.7) is equal at leading order to the reflected wave amplitude determined by the modified mild-slope equation, the key feature of the revised mild-slope equation.

It is evident that the functions $f^{(1)}$ and $f^{(2)}$ will generally give rise to different values $|R^{(1)}|$ and $|R^{(2)}|$ at leading order for every topography and not just those depth functions $h(x)$ that bring into play the deficiencies in the standard mild-slope equation identified in the introduction. Previous computations for a limited number of bedforms have shown discrepancies between the scattered wave amplitudes determined by (1.5) and (1.7) and it is now clear that they are not untypical.

To assess the effects of $f^{(1)}$ and $f^{(2)}$, we first note that $f^{(2)}(K) > 0$ is a decreasing function whereas, as K increases, $f^{(1)}(K)$ changes sign from positive to negative at $K = \mathscr{K}_0 \approx 2.40$, where $(K/2)\tanh(K/2) = 1$ (that is, where $kh\tanh(kh) = 1$) and has a minimum in $K > \mathscr{K}_0$. It is easily checked that

$$F(K) = f^{(1)}(K)/f^{(2)}(K) = 1 - (K\cosh K - \sinh K)/(K + \sinh K) < 1$$

for $K > 0$, with $F(0) = 1$. Indeed, $F(K)$ is a decreasing function, with $F(K) \to -\infty$ as $K \to \infty$. Clearly, $F(\mathscr{K}_0) = 0$ and a straightforward calculation shows that $F(\mathscr{K}_1) = -1$, where $\mathscr{K}_1 \approx 3.44$. Therefore, $0 \leqslant |F(K)| \leqslant 1$ for $0 \leqslant K \leqslant \mathscr{K}_1$, the upper bound being attained at both $K = 0$ (the shallow-water limit) and $K = \mathscr{K}_1$, and the lower bound at $K = \mathscr{K}_0$. For $K > \mathscr{K}_1$, $|F(K)| > 1$.

To give concrete illustrations of the consequences of these observations, we consider the two components of the general topography (1.3) separately.

3.3. *Small-amplitude perturbations*

Suppose that $h_1 = h_0$ and $h(x) = h_0(1 + \epsilon\delta(x))$ for $0 \leqslant x \leqslant \ell$, representing small-amplitude perturbations about an otherwise horizontal bed. It follows from (3.14) and (3.15) that

$$\beta_1^{(i)}(\ell) = \epsilon Q^{(i)} + O(\epsilon^2), \quad Q^{(i)} = -\tfrac{1}{2}\epsilon K_0 f^{(i)}(K_0) \int_0^\ell \delta'(x) e^{2ik_0 x}\, dx,$$

where $K_0 = 2k_0h_0$. Therefore (3.13) gives

$$\left|R^{(i)}\right|^2 = \frac{\epsilon^2 \left|Q^{(i)}\right|^2 + O(\epsilon^3)}{1 + \epsilon^2 \left|Q^{(i)}\right|^2 + O(\epsilon^3)} \tag{3.16}$$

and

$$\left|R^{(1)}\right| / \left|R^{(2)}\right| = \left|Q^{(1)}\right| / \left|Q^{(2)}\right| + O(\epsilon) = |F(K_0)| + O(\epsilon). \tag{3.17}$$

Thus, at leading order, this ratio of the reflected wave amplitudes can take any non-negative value, including zero at $K_0 = \mathscr{K}_0$ and unity at $K_0 = \mathscr{K}_1$, depending on the value of K_0, which is determined from (1.2) by the dimensionless depth νh_0. The standard mild-slope equation (1.5) therefore underestimates the reflected wave amplitude for $0 < K < \mathscr{K}_1$, and overestimates it for $K > \mathscr{K}_1$. Although reflection is insignificant for large values of K, $kh = \mathscr{K}_0/2$ and $kh = \mathscr{K}_1/2$ are in the $O(1)$ dimensionless wavenumber range for which the mild-slope equation was devised.

In the particular case of a so-called ripple bed with $\delta(x) = -\sin(\kappa x)$, we readily find that

$$\left|Q^{(i)}\right| = K_0 f^{(i)}(K_0)\gamma \sin(N\pi\gamma)/(\gamma^2 - 1),$$

where $\gamma = 2k_0/\kappa$ and $N = \kappa\ell/2\pi$ is the number of ripples. A more detailed examination shows that the correction terms in (3.16) are $O(\epsilon^4)$ for this topography. The phenomenon of Bragg resonance is evident in the limit $|Q^{(i)}| \to K_0|f^{(i)}(K_0)|N\pi/2$ as $\gamma \to 1$. Using the data of Davies & Heathershaw (1984), with $\epsilon = 0.16$ and $N = 10$ we find that $K_0 = 1.96$. At Bragg resonance (3.16) gives $|R^{(1)}| = 0.28$ and $|R^{(2)}| = 0.67$, values which compare surprisingly well with those obtained by direct computations (given in figure 1), since $|Q^{(i)}| \sim N$ is an extreme case for the present approach. The deduction of (3.17) from (3.16) is compromised by the presence of N, however, as $|R^{(1)}|/|R^{(2)}| = 0.43$ is given by the former equation and $|F(K_0)| = 0.33$.

3.4. *Slowly varying topography*

In this case we let $h(x) = H(\epsilon x)$ in (3.15) and, integrating by parts, we find that

$$\beta_1^{(i)}(\ell) = \frac{i}{2}\left\{ f^{(i)}(K_1)h'(\ell-)e^{i\theta} - f^{(i)}(K_0)h'(0+)\right\} + O(\epsilon^2), \quad \theta = 2\int_0^\ell k(h(s))\, ds,$$

where $K_0 = 2k_0h_0$, as before, and $K_1 = 2k_1h_1$. Approximations to the reflected wave amplitude follow from (3.13) in the form (3.16), but now with

$$\epsilon^2 \left|Q^{(i)}\right|^2 = P_0^{(i)2} + P_1^{(i)2} - 2P_0^{(i)}P_1^{(i)}\cos\theta, \tag{3.18}$$

where $P_0^{(i)} = f^{(i)}(K_0)h'(0+)/2$ and $P_1^{(i)} = f^{(i)}(K_1)h'(\ell-)/2$. If the bed slope is continuous at $x = 0$ and therefore $h'(0+) = 0$, a second integration by parts gives $P_0^{(i)} = f^{(i)}(K_0)h''(0+)/4k_0$. Similarly, $h'(\ell-) = 0$ gives $P_1^{(i)} = f^{(i)}(K_1)h''(\ell-)/4k_1$ and if $h'(0+) = h'(\ell-) = 0$ the correction terms in (3.16) have to be reset to $O(\epsilon^5)$.

Although it is clear that (3.18) will give values of $|R^{(1)}|$ and $|R^{(2)}|$ that are distinct for most $h(x)$ and most values of $\nu\ell$, K_0 and K_1, because of the derived properties of $f(K)$, the relationship between the two amplitudes is obscure for this bedform because of the multiplicity of parameters involved. Some special cases can be identified, however. If $h'(\ell-) = 0$ and $h'(0+) \neq 0$, (3.17) applies and if $h'(0+) = 0$ with $h'(\ell-) \neq 0$ it

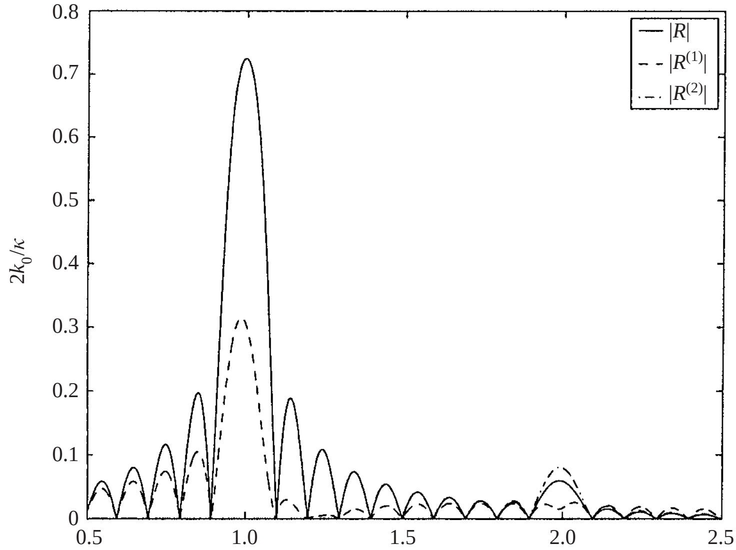

FIGURE 1. Comparison of the reflection coefficients for the periodic bedform described in the text. $|R|$ corresponds to the modified mild-slope equation, $|R^{(1)}|$ to the standard mild-slope equation and $|R^{(2)}|$ to the revised mild-slope equation.

applies again but with K_1 replacing K_0. Also, one of the terms $|Q^{(i)}|$ may vanish and then the two reflected amplitudes are of different orders. For example, if $\theta = 0$ and $h'(0+)f^{(1)}(K_0) = h'(\ell-)f^{(1)}(K_1)$ then $|Q^{(1)}| = 0$ and $|R^{(1)}| = O(\epsilon^2)$. Unless $K_0 = K_1$ then $F(K_0) \neq F(K_1)$ which implies that $f^{(1)}(K_1)f^{(2)}(K_0) \neq f^{(2)}(K_1)f^{(1)}(K_0)$, whence $|Q^{(2)}| \neq 0$ and $|R^{(2)}| = O(\epsilon)$.

A more general inspection of $|R^{(1)}/R^{(2)}|$ using the properties of $f^{(1)}$, $f^{(2)}$ and F shows that the dominant term in this quotient can take any non-negative value, confirming that (1.5) and (2.7) give scattered wave amplitudes that differ at $O(\epsilon)$ for any bedform, except at isolated parameter values.

3.5. *Numerical results*

The preceding analysis is confirmed by numerical results representing each of the two classes of bedforms considered above. The topographies selected for this purpose are those that have been widely used to validate approximate models of two-dimensional scattering.

Scattering by small perturbations in the topography is illustrated by the periodic bed referred to earlier, for which $h(x) = h_0(1 - \epsilon \sin(\kappa x))$ $(0 \leqslant x \leqslant \ell)$ with $\epsilon = 0.16$ and $N = 10$. As comparison with the experimental data produced by Davies & Heathershaw (1984) has been carried out previously for the mild-slope and modified mild-slope equations (in Chamberlain & Porter 1995, for example), it is not duplicated here and the computational results presented in figure 1 are confined to the present investigation. The three versions $|R|$, $|R^{(1)}|$ and $|R^{(2)}|$ of the magnitude of the reflected wave amplitude that are plotted correspond respectively to the modified mild-slope equation (1.7) or (2.4), the original mild-slope equation (1.5) and the new mild-slope equation (2.7); the abscissa is $2k_0/\kappa$, where k_0 is the incident wavenumber.

Graphs of the same quantities are shown in figure 2 in the case of the slowly varying topography consisting of a linear ramp joining the depths h_0 and $h_1 = h_0/3$ and given by $h(x) = h_0(1 - 2x/3\ell)$ $(0 \leqslant x \leqslant \ell)$. The parameter in this case is the dimensionless

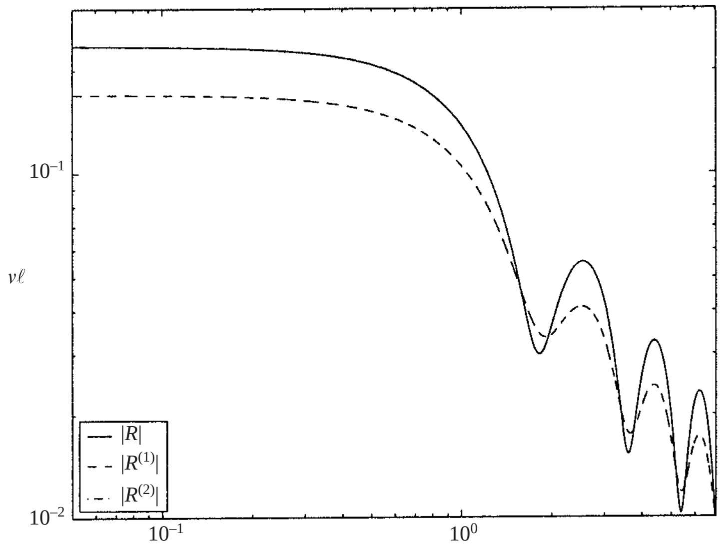

FIGURE 2. Comparison of the reflection coefficients for the ramp bedform described in the text. $|R|$ corresponds to the modified mild-slope equation, $|R^{(1)}|$ to the standard mild-slope equation and $|R^{(2)}|$ to the revised mild-slope equation.

ramp length $\nu\ell$. Porter & Staziker (1995) compared the scattering characteristics predicted by the mild-slope and modified mild-slope equations for this bedform with those obtained using full linear theory by Booij (1983).

Both figures confirm that $|R^{(1)}|$ and $|R^{(2)}|$ differ except at isolated parameter values, as we have established analytically for a general topography. Moreover, the curves representing $|R|$ and $|R^{(2)}|$ are almost indistinguishable at the resolution in the figures, except near $k_0 = \kappa$ in figure 1. Similar discrepancies have previously been noted at this location, evidently because the approximations are unable to resolve the weak secondary resonance there.

The near coincidence of $|R|$ and $|R^{(2)}|$ in these examples suggests that the coefficient $v(h)$ of the second-order correction term in (2.4) should be investigated more closely. It is not difficult show and confirm computationally that this coefficient has a maximum value of $v(h) \approx 0.030$ at $K \approx 3.254$ (corresponding to $\nu h \approx 1.506$). Therefore the new mild-slope equation gives virtually the same values of the reflected wave amplitude as the full modified version, for every topography.

4. Conclusions

A new form of the mild-slope equation has been derived which gives the leading term in the solution of the more accurate modified mild-slope equation. This corrects an inconsistency in the original mild-slope equation which produces solutions differing at lowest order in the mild-slope parameter from those of the full equation. Two types of bedform can be readily identified as giving rise to this discrepancy: those having a rapidly varying small-amplitude component and those having slope discontinuities. However, an analytic solution developed for two-dimensional wave scattering has shown that a discrepancy actually occurs for every topography.

In the process of devising the new mild-slope equation, an alternative simpler representation of the modified mild-slope equation has been identified that has several advantages over the existing form. Further, the correction term in new version of the modified equation, which is second order in the bed slope and also has a small coefficient, has virtually no effect on the scattered wave amplitude. The new mild-slope equation is therefore not only simpler than its predecessor but is almost as accurate as the modified mild-slope equation, which is effectively redundant.

Extensions of the mild-slope equation may benefit from a corresponding analysis to that carried out here. Massel (1993) and Porter & Staziker (1995) enhanced the approximation (1.4) so as to include N terms deriving from evanescent wave modes. Subsequently, Athanassoulis & Belibassakis (1999) showed that this approach is defective in the sense that the exact solution cannot be attained in the limit $N \to \infty$. A related issue is that the average wave power is not conserved across varying topography in the two earlier models. Athanassoulis & Belibassakis (1999) remedied these defects by the addition of a further mode to those used by the previous authors. Inevitably, the convergence of an approximation derived from a variational principle is improved by the use of a trial space having more of the properties of the exact solution, and this is the practical benefit of the improvement given by Athanassoulis & Belibassakis (1999).

Although the inclusion of higher-order modes in the approximation process leads to additional accuracy, the resulting system of coupled differential equations is unwieldy and has to be solved subject to a set of coupled jump conditions holding across discontinuities in the bed slope. It is possible that the basic idea used in the present work can be carried over to the multi-mode cases, transforming the coupled equations into a more concise form which can be simplified without discarding any essential features and removing the jump discontinuities from the numerical solution procedure.

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